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THE PIECEWISE LINEAR UNKNOTTING OF CONES

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§1. INTRODUCTION

A POLYHEDRON X is said to unknot in another polyhedron Y, if any two piecewise linear embeddings of X and Y that are homotopic are also ambient isotopic; this means that one embedding can be carried to the other by a piecewise linear isotopy of Y. If Y happens to be a polyhedral sphere or ball, this definition can be simplified by reference to the result of Alexander and Gugenheim [3], that any piecewise linear orientation preserving homeomorphism of Y is, in this case, isotopic to the identity. The theory of unknotting when X and Y are both manifolds has recently been much studied, taking its impetus from the theorems of Zeeman [9]. These show that S^p, the p-dimensional sphere, unknots in S^n, and that proper embeddings (i.e. with boundary embedded in boundary) of the p-ball B^p, in B^n, also unknot, provided that in both cases, $(n-p) \geqq 3$. The apparatus developed by Zeeman for the proof of these results is here examined and extended. The main theorem (Theorem (1)) obtained in this paper is that if (B^n, X^p) is a pair consisting of a ball B^n, with a cone X embedded in it, with the base of the cone, and only the base of the cone, embedded in the boundary of the ball, then (B^n, X^p) is pairwise homeomorphic to the cone on $(\partial B^n, \partial B^n \cap X^p)$, by a homeomorphism that is the identity on ∂B^n, this being subject to the usual proviso that $(n-p) \geqq 3$. This result leads immediately to theorem 5 which is a suspension theorem for the property of "being unknotted in a sphere". More precisely it is shown that if X^p unknots in S^n, $(n-p) \geqq 3$, then the r-fold suspension of X unknots in S^{n+r}.

In geometrically flavoured topology, the ideas of cones and suspensions seem to be at home mostly in the combinatorial category. The fact that Zeeman's methods [9] generalise to give results on arbitrary cones and suspensions emphasises that in his original results, on the unknotting of S^p in S^n, the important property, required of the sphere S^p, is not really that S^p is a manifold, but that S^p is the suspension of something simpler, namely the sphere S^{p-1}. As the notion of suspension is rather unnatural in the theory of differential topology, it is not too surprising that very different results have been obtained by Haefliger [4] for the possibility of differentiably unknotting spheres in spheres of higher dimensions.

The proof of Theorem (1) is by induction on the dimension of the ball B^n, and it rests heavily on the rather technical Proposition (1), and on the generalisation of the Stallings–Zeeman sunny collapsing process. Theorems (1) and (5) are, however, not simply elegant results in themselves; they can be used, together with Proposition (1), to give neat and possibly surprising corollaries. One such result is that (Theorem (6)) two embeddings of a polyhedron X^p in S^n, $n - p \geqq 3$, are ambient isotopic if there exists a homeomorphism of $X^p \times I$ into $S^n \times I$ which, when restricted to $X \times 0$ and $X \times 1$, reduces to the two given embeddings. This, in the case of isotopies in S^n, improves on a theorem of Hudson [5] which requires X to be a manifold, and the embedding of $X^p \times I$ in $S^n \times I$ to be level preserving, before the two embeddings can be proved ambient isotopic. Theorem (7) follows from this result. It is there shown that if S^n, the boundary of B^{n+1}, contains two disjoint spheres S^p and S^q, with $n - p \geqq 3$, and $n - q \geqq 3$, then S^p and S^q are unlinked in S^n, if and only if they bound disjoint balls (of dimensions $p + 1$ and $q + 1$ respectively) in B^{n+1}. It is perhaps worth noting that this is not true if the codimension, $n - p$, is two. The differential analogue of this is true, however, in codimensions $\geqq 3$ by Smale's work [7].

It is shown that the polyhedron consisting of two spheres with a point in common unknots in another sphere of codimension $\geqq 3$. Also it is proved that if two embeddings of B^p in S^n, $n - p \geqq 3$, agree on the boundary of B^p, (the two images may intersect in any way), then there is an isotopy of S^n which moves one embedding to the other, but keeps the embedding of the boundary of B^p fixed throughout.

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§2. DEFINITIONS AND PRINCIPAL RESULTS

Throughout this section we shall be working within the category of polyhedra and piecewise linear embeddings. This category can be described in the following way: A *triangulation* of a topological space X is a homeomorphism of a finite simplicial complex onto X. Two such triangulations, $f : K \longrightarrow X$ and $g : L \longrightarrow X$ are said to be piecewise linearly *related* if $g^{-1}f$ is a piecewise linear homeomorphism from K to L. A set of triangulations, which are piecewise linearly related, is said to be *maximal* if any triangulation of X, that is piecewise linearly related to any triangulation of this set, belongs to the set. A *polyhedron* X is, then, a topological space together with a maximal set of piecewise linearly related triangulations. If $f : K \longrightarrow X$ and $g : L \longrightarrow Y$ are triangulations of the polyhedra X and Y, a map $\phi : X \longrightarrow Y$ is called piecewise linear if $g^{-1}\phi f : K \longrightarrow L$ is piecewise linear; this definition is independent of the choice of f and g from the triangulations of the given structures of the polyhedra. Y is a *sub-polyhedron* of a polyhedron X if for some triangulation $f : K \longrightarrow X$ of X, $Y = fL$, for some subcomplex L of $K; f|L : L \longrightarrow Y$ induces the polyhedral structure of Y.

We shall use the symbols B^n and S^n to denote n-dimensional polyhedral balls and spheres respectively. Thus B^n is a polyhedron that has a triangulation $f : \Delta^n \longrightarrow B^n$, and S^n a triangulation $g : \partial\Delta^{n+1} \longrightarrow S^n$. ($\Delta^n$ is the n-simplex, and ∂ is used to denote a boundary). All embeddings and homeomorphisms of polyhedra are to be understood to be piecewise linear, unless otherwise stated.

If M and L are polyhedra, and L is a sub-polyhedron of M, we shall refer to them as a *pair* (M, L). The *codimension* of the pair is $\dim.M - \dim.L$. Two pairs (M_1, L_1) and (M_2, L_2) are homeomorphic if there is a homeomorphism from M_1 to M_2 which, when restricted to L_1, is a homeomorphism from L_1 to L_2. (R, S) is called a *sub-pair* of the pair (M, L), if R is sub-polyhedron of M, and $R \cap L = S$.

The join of two polyhedra X and Y will be written XY, or if it seems that confusion might arise, as $X * Y$. XY has a uniquely defined structure as a polyhedron (see Zeeman [10]). In particular, if a is a single point, aX is the *cone* on X, and X regarded as a sub-polyhedron of aX is called the *base* of the cone. Joins of a single polyhedron to pairs of polyhedra will occur, and $X*(M, L)$ will denote the pair (XM, XL), and similarly $a*(M, L)$ denotes $aM, aL)$. The join of two distinct points to a polyhedron X is the *suspension* of X, written as ΣX, and $\Sigma^r X$ denotes the r-fold suspension of X.

DEFINITION. *If X is any polyhedron, and aX the cone on X, an embedding $f : aX \longrightarrow B^n$ is called* proper *if $f^{-1}\partial B^n = X$.*

This simply means that a proper embedding of a cone embeds the base of the cone in the boundary of the ball, and the remainder of the cone in the interior of the ball. It must be emphasised that, subject to this condition (and that of piecewise linearity), the embedding may be quite arbitrary. A simple example of a proper embedding of a cone on a ball is an embedding of B^p in B^n, which embeds ∂B^p in ∂B^n and $\mathring{B}^p$ in $\mathring{B}^n$. If $n - p = 2$, this may be a knotted embedding but it is still proper. Another example is the inclusion of aX in aS^{n-1}, where X is a sub-polyhedron of S^{n-1}. In this second case the embedding is rather a special one, for aX is embedded as a sub-cone of B^n regarded as the cone aS^{n-1}.

An *isotopy* of a polyhedron Y is a level preserving homeomorphism $h : I \times Y \longrightarrow I \times Y$, such that if $h_t : Y \longrightarrow Y$ is defined for all $t \in I$ by $h(t, y) = (t, h_t y)$, then h_0 is the identity. We shall sometimes also refer to the set of homeomorphisms h_t as an isotopy. A point $y \in Y$ is said to be *kept fixed* by the isotopy if $h_t y = Y$ for all $t \in I$. Two embeddings f and g of X into Y are *ambient isotopic* if there exists an isotopy of Y, such that $h_1 f = g$. X is said to *unknot* in Y if any two homotopic embeddings of X in Y are ambient isotopic.

If $f : aX \longrightarrow B^n$ is a proper embedding of the cone aX in B^n, from the above definition $f|X$ is an embedding of X in ∂B^n. We can define an associated embedding $\bar{f} : aX \longrightarrow b * \partial B^n$ by defining $\bar{f}a = b$, and extending $f|X$ conically. We call $\bar{f}$ the conical *extension* of $f|X$. We can think of $b * \partial B^n$ as being equivalent to B^n, and of f and $\bar{f}$ as being two proper embeddings of aX in B^n. Then, Theorem (1) says that, for codimension $\geqq 3$, these two embeddings are equivalent, and implies that we can, isotopically, slide the embedding f to the nice, straight, sub-cone embedding $\bar{f}$, leaving the boundary of B^n fixed in the isotopy. More precisely we shall prove:

THEOREM (1). *Let $f : aX \longrightarrow B^n$ be a proper embedding of aX, where* $\dim aX \leqq n - 3$, *then there is a homeomorphism $h : B^n \longrightarrow b * \partial B^n$ such that $h|\partial B^n$ is the identity, and the following diagram is commutative (where $\bar{f}$ is the conical extension of $f|X$):*

$$\begin{tikzcd} aX \arrow[r, "f"] \arrow[dr, "\bar{f}"'] & B^n \arrow[d, "h"] \\ & b * \partial B^n \end{tikzcd}$$

To convert this result into a statement concerning isotopy and unknotting we need the following lemmas, due essentially to Alexander [1] and Gugenheim [3]. An outline of the proofs of them is given here.

LEMMA (1). *Let X be a sub-polyhedron of S^{n-1}, and let h be any homeomorphism, $h : bS^{n-1} \longrightarrow bS^{n-1}$ such that $h|(S^{n-1} \cup bX) = 1$. Then h is isotopic to the identity by means of an isotopy which keeps $(S^{n-1} \cup bX)$ fixed.*

Proof. We construct the isotopy $H : I \times bS^{n-1} \longrightarrow I \times bS^{n-1}$ as follows.

$$H(t, x) = \begin{cases} (t, x) & \text{if } t = 0 \quad \text{or if } x \in S^{n-1} \\ (t, hx) & \text{if } t = 1 \end{cases}$$

This defines H level preserving on $\partial(I \times bS^{n-1})$. Define H level preserving on $I \times bS^{n-1}$ by mapping a selected point of the interior of $I \times b$ to itself, and joining linearly to the boundary of $I \times bS^{n-1}$. H is then the desired isotopy.

LEMMA (2). *If $H : I \times S^{n-1} \longrightarrow I \times S^{n-1}$ is an isotopy of S^{n-1} that keeps $X \subset S^{n-1}$ fixed, then H can be extended to an isotopy $\bar{H} : I \times bS^{n-1} \longrightarrow I \times bS^{n-1}$, which keeps bX fixed.*

Proof. Define $\bar{H}|0 \times bS^{n-1}$ as the identity and $\bar{H}|1 \times bS^{n-1}$ as the conical extension of $H|1 \times S^{n-1}$. $\bar{H}$ is then defined level preserving on $I \times bS^{n-1}$ by mapping a selected point of the interior of $I \times b$ to itself and joining linearly to the boundary of $I \times bS^{n-1}$.

If bX is vacuous, the above results reduce to the usual classical lemmas, and lead to the following corollary in the usual way (see for example [3] or [10]).

COROLLARY. *Any orientation preserving homeomorphism of S^n to itself is isotopic to the identity.*

The proof of Theorem (1) will be deferred to a later section, but using Lemmas (1) and (2), we can at once deduce from the theorem the following results:

THEOREM (2). *If f and g are proper embeddings of aX in B^n, with* $\dim aX \leqq n - 3$, *and $f|(X \cup aY) = g|(X \cup aY)$ for some polyhedron Y contained in X, then f and g are ambient isotopic, and $\partial B^n \cup f(aY)$ can be kept fixed in the isotopy.*

Proof. By Theorem (1) there are homeomorphisms h_1 and h_2 where $h_i : B^n \longrightarrow b * \partial B^n$; h_1 and h_2 are the identity on ∂B^n, and

$$h_1 f = \bar{f} = \bar{g} = h_2 g.$$

Then, $h_2 h_1^{-1}$ is a homeomorphism of $b * \partial B^n$ to itself that is the identity on $\partial B^n \cup b * fY$. By Lemma (1) this homeomorphism is isotopic to the identity by an isotopy H_t which is fixed (for all $t \in I$) on $\partial B^n \cup b * fY$. Thus, $h_2^{-1}H_th_1$ is an isotopy of B^n that is fixed on $\partial B^n \cup f(aY)$, and provides an ambient isotopy sending embedding g to embedding f.

COROLLARY. *If f and g are proper embeddings of aX in B^n, $\dim, aX \leqq n-3$, and if there is a homeomorphism $h : \partial B^n \longrightarrow \partial B^n$, such that $hf|X = g|X$, then h extends to $\bar{h} : B^n \longrightarrow B^n$ such that $\bar{h}f = g$.*

Proof. As B^n is homeomorphic to $b * S^{n-1}$, any homeomorphism of ∂B^n to itself can be extended (cone-wise in $b * S^{n-1}$) to some homeomorphism of B^n. Let θ be any such extension of h. By the theorem there is a homeomorphism ϕ of B^n, fixed on ∂B^n, such that $\phi\theta f = g$. As $\phi\theta|\partial B^n = h$, $\phi\theta$ is an extension of h of the required form.

Theorem (2) is perhaps most easily described as a relative unknotting theorem for properly embedded cones: Any two such embeddings which agree on the base of the cone are ambient isotopic, and if the embeddings agree on a sub-cone, this can be kept fixed in the isotopy. If it is assumed that the base of the cone unknots in the boundary of the ball, then the next theorem shows that proper embeddings of the cone in the ball unknot in codimension three or more (i.e. any two proper embeddings are ambient isotopic):

THEOREM (3). *If $n-p \geqq 3$, and X^{p-1} unknots in S^{n-1}, then any two proper embeddings of aX in B^n are ambient isotopic.*

Proof. Let f and g be proper embeddings of aX in B^n. Since X unknots in S^{n-1}, there is an isotopy $H_t : \partial B^n \longrightarrow \partial B^n$ such that H_0 is the identity and $H_1 f|X = g|X$. This isotopy extends to an isotopy $\bar{H}_t$ of B^n by Lemma (2). Then $\bar{H}_1 f$ and g are proper embeddings of aX in B^n that agree on X. By Theorem (2), $\bar{H}_1 f$ and g are ambient isotopic, and since ambient isotopy is an equivalence relationship, f and g are ambient isotopic.

We prove the following theorem by a similar argument to that used in the proof of Theorem (3), but in this case we make use of the property that when proper embeddings agree on a sub-cone, this sub-cone can be kept fixed in an ambient isotopy between the two embeddings:

THEOREM (4). *If f and g are embeddings of B^p in S^n, $n-p \geqq 3$, and $f|\partial B^p = g|\partial B^p$, then f and g are ambient isotopic, and $f(\partial B^p)$ can be kept fixed in the isotopy.*

Proof. The theorem is trivially true if $n = 3$, as B^p is then either empty, or consists of a single point. Thus, inductively, we assume the theorem to be true for any pair of embeddings of B^q in S^m, $m - q \geqq 3$, which agree on ∂B^q, provided that $m < n$. Now we suppose that f and g are given as in the statement of the theorem. Let L_1 and L_2 be triangulations of B^p, and K a triangulation of S^n, such that $f : L_1 \longrightarrow K$ and $g : L_2 \longrightarrow K$ are simplicial. Let v be a vertex of $f(\partial L_1)''$ where $(\partial L_1)''$ is a second derived of ∂L_1, and let $f^{-1}v = a_1$, $g^{-1}v = a_2$. Then star (v, K'') is a combinatorial ball, (stars of vertices in complexes are always taken as closed stars), and the closure of $K''-$ star (v, K'') is another ball, and these two balls have in common the combinatorial sphere link (v, K''). Similarly star (a_1, L_1'') and the closure of L_1-star (a_1, L_1'') are balls having the $(p-1)$-dimensional ball, link(a_1, L_1''), in common. A similar situation occurs in L_2.

Returning to the polyhedra B^p and S^n, we infer from the above discussion that S^n is equal to the union of two polyhedral balls B_1^n and B_2^n with their boundary S^{n-1} in common. Similarly B^p can be considered as the union of two p-balls, one mapped by f into B_1^n, and the other into B_2^n. We express this in the following way: There is a homeomorphism $f_1 : (c_1 \cup c_2) * B^{p-1} \longrightarrow B^p$ such that $ff_1 : c_i * B^{p-1} \longrightarrow B_n^i$ is a proper embedding of the cone on B^{p-1}, for $i = 1, 2$. There is a homeomorphism $g_1 : (c_1 \cup c_2) * B^{p-1} \longrightarrow B^p$ with an analogous property, and

$$f_1|(c_1 \cup c_2) * \partial B^{p-1} = g_1|(c_1 \cup c_2) * \partial B^{p-1}.$$

By Theorem (1) applied to B_i^n and B_2^n, there is a homeomorphism $\phi : S^n \longrightarrow (b_1 \cup b_2) * S^{n-1}$ such that ϕ is the identity on S^{n-1}, and

$$\phi ff_1(c_i * \partial B^{p-1}) = b_i * ff_1 \partial B^{p-1}, \qquad i = 1, 2.$$

Now ff_1 and gg_1 give embeddings of B^{p-1} in S^{n-1} which agree on ∂B^{p-1}. By the induction hypothesis there is an isotopy of S^{n-1} which leaves $ff_1 \partial B^{p-1}$ fixed, and which sends $ff_1|B^{p-1}$ to $gg_1|B^{p-1}$. This can, by Lemma (2), be extended to an isotopy H_t of $(b_1 \cup b_2) * S^{n-1}$, which keeps $\phi ff_1(c_i * \partial B^{p-1})$ fixed for $i = 1, 2$. $H_1 \phi ff_1$ and $H_1 \phi gg_1$ give proper embeddings of $(c_i * B^{p-1})$ in $b_i * S^{n-1}$, which agree on B^{p-1} and on $c_i * \partial B^{p-1}$. By Theorem (2) there is an isotopy G_t of $(b_1 \cup b_2) * S^{n-1}$, which is fixed on S^{n-1} and on $\phi ff_1(c_i * \partial B^{p-1})$ and which sends $H_1 \phi ff_1$ to $H_1 \phi gg_1$. Thus $\phi^{-1} G_t H_t \phi$ is an isotopy of S^n, which provides an ambient isotopy from ff_1 to gg_1, and keeps $f \partial B^p$ fixed, and so $\phi^{-1} G_1 H_1 \phi f = gg_1 f_1^{-1}$. However, $g_1 f_1^{-1} : B^p \longrightarrow B^p$ is fixed on ∂B^p, and is by Lemma (1) isotopic to the identity keeping ∂B^p fixed. There is then an isotopy F_t of S^n which leaves $g(\partial B^n)$ fixed, and which sends $gg_1 f_1^{-1}$ to g [6, Corollary (2.4)]. Thus finally $F_t \phi^{-1} G_t H_t \phi$ is the required isotopy of S^n, sending f to g. This completes the induction argument and the theorem is proved.

We now state and prove the suspension stability theorem for the property of unknotting in a sphere.

THEOREM (5). *If $n - p \geqq 3$ and X^{p-1} unknots in S^{n-1}, then ΣX unknots in S^n.*

Proof. Let f and g be two embeddings of ΣX in S^n, and consider ΣX as $(a_1 \cup a_2) * X$. A regular neighbourhood of $f(\Sigma X)$ mod $f(a_2 X)$ in S^n is a polyhedral n-ball P_1, such that $f|a_1 X$ is a proper embedding of $a_1 X$ in P_1 [6], and if $P_2 = \text{closure } (S^n - P_1)$, $f|a_2 X$ is a proper embedding of $a_2 X$ in P_2. Let $i : X \longrightarrow S^{n-1}$ be some selected standard embedding of X in S^{n-1}, and let $\bar{\imath}$ be the suspension of this embedding, $\bar{\imath} : (a_1 \cup a_2) * X \longrightarrow (b_1 \cup b_2) * S^{n-1}$. Since X unknots in S^{n-1} there is a homeomorphism $\phi : \partial P_1 \longrightarrow S^{n-1}$, such that $\phi f|X = i$, by the corollary to Theorem (2), this extends to $\bar{\phi} : S^n \longrightarrow (b_1 \cup b_2) * S^{n-1}$ such that $\bar{\phi} f = \bar{\imath}$. Similarly there exists a homeomorphism $\bar{\psi} : S^n \longrightarrow (b_1 \cup b_2) * S^{n-1}$ such that $\bar{\psi} g = \bar{\imath}$. Then $\bar{\psi}^{-1} \bar{\phi} f = g$, and, since by the corollary to Lemma (2), $\bar{\psi}^{-1} \bar{\phi}$ is isotopic to the identity, f and g are ambient isotopic.

COROLLARY (1). *If $n - p \geqq 3$ and X^p unknots in S^n, then $\Sigma^r X^p$ unknots in S^{n+r}.*

Proof. This follows at once by iteration of the theorem.

COROLLARY (2). *If $n - p \geqq 3$, S^p unknots in S^n.*

Proof. Take X as a pair of points in S^{n-p}, and suspend p times, using Corollary (1).

This second corollary is Zeeman's result of [9]; it must be emphasised that the proof of Theorem (1), to be given later, depends on a generalisation of Zeeman's methods.

COROLLARY (3) *If $n-p \geqq 3$, the polyhedron consisting of two p-spheres with a p-ball in common unknots in S^n.*

Proof. Take X as three points, which unknots in S^{n-p}. Suspend p times using Corollary (1).

By taking X as any number of points, Corollary (3) generalises to the fact that the polyhedron consisting of a finite number of p-spheres with a p-ball in common unknots in S^n, if $n-p \geqq 3$. It is interesting to contrast this result with the fact that the polyhedron which consists of two disjoint S^p's in S^n *can* knot when $n-p \geqq 3$; this being simply the phenomenon of linking.

The concept of two embeddings of a polyhedron X in another Y being ambient isotopic has been discussed above. We now state the result which shows that if Y is a polyhedral sphere, and $\dim Y - \dim X \geqq 3$, the condition for two embeddings of X in Y to be ambient isotopic is equivalent to a seemingly much weaker condition:

THEOREM (6). *Let f and g be embedding of the polyhedron X^p in S^n, where $n-p \geqq 3$. Suppose there is an embedding $H: X \times I \longrightarrow S^n \times I$ such that $H^{-1}(S^n \times 0) = X \times 0$, $H^{-1}(S^n \times 1) = X \times 1$, and $H(x, 0) = (f(x), 0)$, $H(x, 1) = (g(x), 1)$, for all $x \in X$, then f and g are ambient isotopic.*

The proof of this theorem will be given in §4, but from Theorem (6) we can deduce the following result:

THEOREM (7). *S^p and S^q contained in S^n, $n-p \geqq 3$, $n-q \geqq 3$, are unlinked if and only if, regarding S^n as ∂B^{n+1}, S^p and S^q bound disjoint balls B^{p+1}, B^{q+1}, whose interiors are contained in the interior of B^{n+1}.*

Proof. S^p and S^q contained in S^n are said to be unlinked if they are contained in disjoint n-balls in S^n. If this is the case, it is easy to show that S^p and S^q bound disjoint balls B^{p+1}, B^{q+1} in B^{n+1}. Thus suppose, conversely, that these balls are given, and it is required to show that S^p and S^q are unlinked. Let $x \in \dot{B}^{p+1}$, the interior of B^{p+1}, and $y \in \dot{B}^{q+1}$. Let $N(x, B^{n+1})$ be a small regular neighbourhood of x in $\dot{B}^{n+1}$ which intersects B^{p+1} in $N(x, B^{p+1})$, a regular neighbourhood of x in $\dot{B}^{p+1}$; let $N(y, B^{n+1})$, $N(y, B^{q+1})$ be similarly defined, so that $N(x, B^{n+1})$ and $N(y, B^{n+1})$ are disjoint. These neighbourhoods are all balls of the relevant dimensions. The boundary of the pair $N(x, B^{n+1})$, $N(x, B^{p+1})$ is a pair of spheres (S^n_α, S^p_α) and we can find a ball B^n_α such that $S^p_\alpha \subset B^n_\alpha \subset S^n_\alpha$; similarly for y, we have the pair (S^n_β, S^q_β) and $S^q_\beta \subset B^n_\beta \subset S^n_\beta$ (see Fig. 1). Let $A \in S^n_\alpha - B^n_\alpha$, and $B \in S^n_\beta - B^n_\beta$. Let π be a simple polyhedral path joining A and B in B^{n+1}, not meeting B^{p+1}, B^{q+1} or $N(x, B^{n+1}) \cup N(y, B^{n+1})$, except at A and B, and let M be a regular neighbourhood of π in $\dot{B}^{n+1} - (B^{p+1} \cup B^{q+1} \cup B^n_\alpha \cup B^n_\beta \cup \dot{N}(x, B^{n+1}) \cup \dot{N}(y, B^{n+1}))$ which meets $S^n_\alpha \cup S^n_\beta$ regularly, [6]. M is an $(n+1)$-ball containing π, and M meets S^n_α and S^n_β each in an n-ball contained in ∂M. Then $N(x, B^{n+1}) \cup M \cup N(y, B^{n+1})$ is an $(n+1)$-ball B^{n+1}_1 with boundary S^n_1. But $B^n_\alpha \subset S^n_1$ and $B^n_\beta \subset S^n_1$, thus S^p_α and S^q_β are unlinked in S^n_1. Now $B^{n+1} - \dot{B}^{n+1}_1$ is homeomorphic to $S^n \times I$, and $B^{p+1} -$

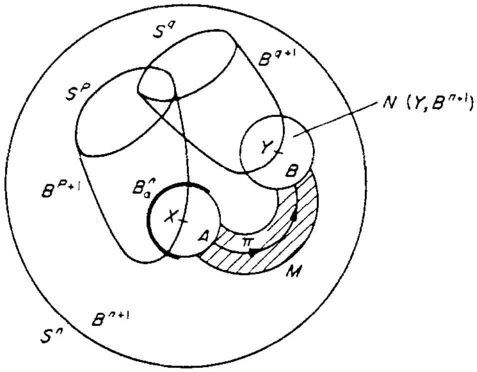

FIG. 1.

$\mathring{N}(x, B^{p+1})$ and $B^{q+1} - \mathring{N}(y, B^{q+1})$ can be regarded as the image of an embedding of $(S^p \cup S^q) \times I$ in $B^{n+1} - \mathring{B}^{n+1}_1$. Theorem (6) implies there is a homeomorphism between the pairs $(S^n, S^p \cup S^q)$ and $(S^n_1, S^p_\alpha \cup S^q_\beta)$, and so the fact that S^p_α and S^q_β are unlinked in S^n_1 implies that S^p and S^q are unlinked in S^n.

Remark (1). The proof of Theorem (7) generalises immediately to show that any number of spheres of codimension $\geqq 3$ in S^n are unlinked if they bound disjoint balls in B^{n+1}.

Remark (2). The result of Theorem (7) (and hence also that of Theorem (6)) is not true in codimension 2. This can be seen by considering the two S^1's in S^3 which link as shown in Fig. 2, level 1. These S^1's bound disjoint B^2's in $S^3 \times I$ and hence also in B^4. A proof of this is indicated by Fig. 2, which shows, in the manner originated by Fox [2], the intersection of the two B^2's with various levels of $S^3 \times I$.

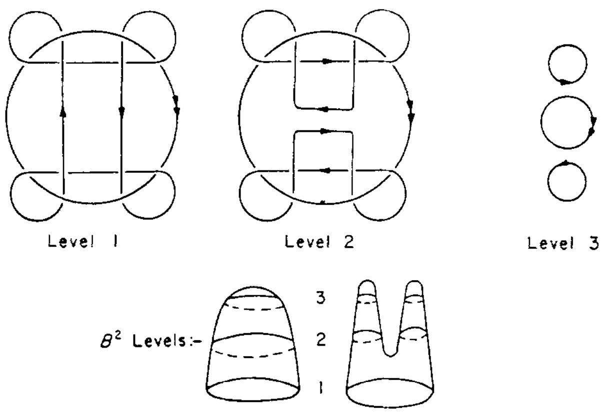

FIG. 2.

We now define $S^p \vee S^q$ as the polyhedron consisting of a p-sphere and a q-sphere with one point v in common.

THEOREM (8). *$S^q \vee S^p$ unknots in S^n if $n - p \geqq 3$, $n - q \geqq 3$.*

Note. Any embedding $f : S^p \vee S^q \longrightarrow S^n$ is locally unknotted in the following sense. Suppose L triangulates $S^p \vee S^q$, K triangulates S^n, and $f : L \longrightarrow K$ is simplicial. Then if $x \in L$, the complex pair $(\text{Link}(fx, K), f\,\text{Link}(x, L))$ is always pairwise homeomorphic to the pair obtained in similar fashion from any second embedding. If $x \neq v$ this follows from the fact that S^{p-1} and S^{q-1} unknot in S^{n-1} (Theorem (5), Corollary (2)), and if $x = v$, link (v, L) is the disjoint union of a combinatorial $(p-1)$-sphere and a $(q-1)$-sphere, which are embedded by f in the $(n-1)$-sphere, link (fx, K). The images under f of these two spheres are unlinked in link (fx, K), by Theorem (7), as they bound disjoint balls in the ball $K\text{-st}\dot{a}\text{r}(fv, K)$.

Proof of Theorem (8). $S^p \vee S^q$ is homeomorphic to X where X is $[(a \cup b) * S^{p-1}] \cup_a [(a \cup c) * S^{q-1}]$. Let f and f_1 be two embeddings of this in S^n. Let P be a regular neighbourhood of $f(bS^{p-1})$ *mod* $\overline{f(X - bS^{p-1})}$ in S^n, and Q a regular neighbourhood of $f(cS^{q-1})$ mod $\overline{f(X - cS^{q-1})}$ in $S^n - P$. Let P_1 and Q_1 be similarly described with respect to f_1. Then, from [6], P is a ball, meeting fX only in $f(bS^{p-1})$, and f gives a proper embedding of bS^{p-1} in P; a similar remark applies to Q, P_1 and Q_1. There is, by Theorem (5), Corollary (2), a homeomorphism $\phi : \partial P \longrightarrow \partial P_1$ such that $\phi f | S^{p-1} = f_1 | S^{p-1}$, which extends by the corollary to Theorem (2) to $\phi : P \longrightarrow P_1$ such that

$$\phi f | bS^{p-1} = f_1 | bS^{p-1}.$$

Similarly there is a homeomorphism $\psi : Q \longrightarrow Q_1$ such that

$$\psi f | cS^{q-1} = f_1 | cS^{q-1}.$$

Let $A \in \partial P$, $B \in \partial Q$ and let π be a simple polyhedral path in S^n connecting A and B, not meeting fX, nor $P \cup Q$ except at A and B. Let M be a regular neighbourhood of π in $S^n - (fX \cup \dot{P} \cup \dot{Q})$, meeting $\partial P \cup \partial Q$ regularly [6]. M is then an n-ball in S^n, $M \cap fX = \varnothing$, and $M \cap P = \partial M \cap \partial P =$ an $(n-1)$-ball, and similarly $M \cap Q = \partial M \cap \partial Q =$ an $(n-1)$-ball (see Fig. 3). Let π_1 be another polyhedral path in S^n connecting ϕA and ψB, and let

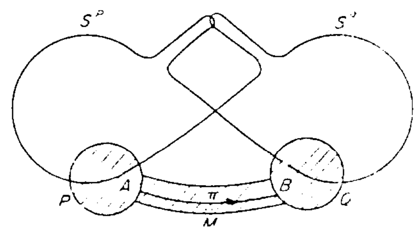

FIG. 3.

M_1 be a regular neighbourhood of π_1 in $S^n - (fX \cup \dot{P}_1 \cup \dot{P}_2)$ such that M_1 has parallel properties to those of M, and $\phi(M \cap P) = M_1 \cap P_1$, $\psi(M \cap Q) = M_1 \cap Q_1$. Now $P \cup M \cup Q$ is an n-ball, and we can extend the homeomorphisms ϕ and ψ to a homeomorphism.

$$\theta : P \cup M \cup Q \longrightarrow P_1 \cup M_1 \cup Q_1.$$

Let B^n be $S^n - (\dot{P} \cup \dot{M} \cup \dot{Q})$ and $B_1^n = S^n - (\dot{P}_1 \cup \dot{M}_1 \cup \dot{Q}_1)$. Then, $f | a * (S^{p-1} \cup S^{q-1})$

gives a proper embedding of $a * (S^{p-1} \cup S^{q-1})$ in B^n, a similar statement is true for f_1, and

$$\theta f|(S^{p-1} \cup S^{q-1}) = f_1|(S^{p-1} \cup S^{q-1}).$$

By the corollary to Theorem (2) θ extends to $\bar{\theta} : S^n \longrightarrow S^n$ such that $\bar{\theta} f = f_1$. f and f_1 are thus ambient isotopic by the corollary to Lemma (2), and the proof is complete.

Theorem (4) and Theorem (5), Corollary (2), combined together show at once, that B^r unknots in S^n provided $(n - r) \geqq 3$. This implies that, if we define $S^p \underset{B^r}{\cup} S^q$ as the polyhedron consisting of an S^p and an S^q with a B^r in common, $S^p \underset{B^r}{\cup} S^q$ is a uniquely defined polyhedron (i.e. independent of the r-balls chosen for the identification process) up to homeomorphism, if $p - r \geqq 3$, $q - r \geqq 3$. Then, applying Theorem (5), Corollary (1), to the result of Theorem (8) we have the following result:

THEOREM (9). *$S^p \underset{B^r}{\cup} S^q$ unknots in S^n if $r + 3 \leqq {}^p_q \leqq n - 3$.*

Remark (1). If in Theorem (8) we had considered the union of S^p and S^q with two points in common, as X, instead of $S^p \vee S^q$, the situation would have been very different. For consider S^{p-1} and S^{q-1} linked in S^{n-1}; the suspension of this situation gives an embedding of X in S^n, which is never homeomorphic to the embedding obtained by suspending S^{p-1} unlinked with S^{q-1} in S^{n-1}. In fact, if $n - p \geqq 3$ and $n - q \geqq 3$, use of Theorem (1) shows that the knots of X in S^n are in one to one correspondence with the links of S^{p-1}, S^{q-1} in S^{n-1}. This can then be suspended to give results analogous to Theorem (9).

Remark (2). If X is the polyhedron consisting of $S^p \cup I \cup S^q$ with one end point of I identified with a point of S^p, and the other with a point of S^q, then X can knot in S^n when $\dim X \leqq n - 3$. This is so because for certain values of n, we can embed X so that the S^p and S^q are linked, and this embedding cannot be ambient isotopic to an embedding in which S^p and S^q are unlinked. However, X is, in this case, of the same homotopy type as $S^p \vee S^q$ which does, by Theorem (8), unknot in S^n. Thus for polyhedra, the property of unknotting in S^n is not an invariant of homotopy type.

§3. REGULAR NEIGHBOURHOODS OF PROPERLY EMBEDDED CONES

The last section was complete in itself except for the omission of the proofs of Theorems (1) and (6). Both these proofs will depend on Proposition (1), and the sole purpose of this section is to state and prove this proposition. In principle, the proposition states that if $f : aX \longrightarrow B^n$ is a proper embedding, then the pair (N, faX), where N is a regular neighbourhood of $f(aX)$ in B^n, is homeomorphic, pairwise, to the pair (star(fa, M), star(fa, L)) where M triangulates B^n, and has L as a subcomplex triangulating faX. Before we can prove this proposition it will be necessary to develop a few rather technical lemmas, and to give an inductive statement, Theorem $(1^{(n)})$, of Theorem (1). Theorem $(1^{(n)})$ is trivially true if $n \leqq 3$, so we shall assume Theorem $(1^{(n-1)})$ to be true, use it to prove Proposition (1), which will, in turn, prove Theorem $(1^{(n)})$. The inductive statement of the theorem is as follows (we are still working in the polyhedral category):

THEOREM $(1^{(n)})$. *Let $f : aX \longrightarrow B^m$ be a proper embedding, where* $\dim aX \leqq m - 3$, *and let $\bar{f} : aX \longrightarrow b * \partial B^m$ be the conical extension of $f|X$, then there is a homeomorphism $h : B^m \longrightarrow b * \partial B^m$, such that $hf = \bar{f}$ and $h|\partial B^m$ is the identity, provided that $m \leqq n$.*

Preparatory Lemmas

The idea of a proper embedding of a cone in a ball has been defined. It will be convenient to parallel this definition here, in the case of embeddings of suspensions in spheres in the following manner: Let B_1 and B_2 be two balls of the same dimension with their boundaries identified, so that $B_1 \cup B_2$ is a sphere. Let $\partial B_1 = B\partial_2 = S$. Suppose that, for some polyhedron Z, g is an embedding of $(a_1 \cup a_2) * Z$ in $B_1 \cup B_2$, such that $g(a_iZ) \subset B_i$ and $g^{-1}S = Z$ for $i = 1, 2$. We call g a proper embedding of $(a_1 \cup a_2) * Z$ in $B_1 \cup B_2$. Now $g(Z) \subset S$, so we can define an associated embedding $\bar{g}$, where

$$\bar{g} : (a_1 \cup a_2) * Z \longrightarrow (x_1 \cup x_2) * S$$

be defining $\bar{g}|Z = g|Z$, $\bar{g}a_i = x_i$, and extending linearly. With this notation we prove:

LEMMA (3). *Suppose B_1 and B_2 are $(n-1)$ balls, and $\partial B_1 = \partial B_2 = S$. Suppose that for some polyhedron Z, $\dim Z \leqq n-5$, g is a proper embedding of $(a_1 \cup a_2) * Z$ in $B_1 \cup B_2$. Then Theorem $(1^{(n-1)})$ implies that there is a homeomorphism h,*

$$h : B_1 \cup B_2 \longrightarrow (x_1 \cup x_2) * S,$$

such that $h|S$ is the identity and $hg = \bar{g}$.

Proof. $g|a_1Z$ is a proper embedding of a_1Z in B_1. By Theorem $(1^{(n-1)})$ we can define $h|B_1 : B_1 \longrightarrow x_1S$ so that $h|S$ is the identity and $hg|a_1Z = \bar{g}|a_1Z$. Defining $h|B_2$, similarly, gives a complete definition of h.

If we have a pair of polyhedra, P_1, which is known to be homeomorphic to a pair $c * (S^{n-1}, Y)$, we want to investigate what pairs can be glued to P_1, so that the composite pair is still homeomorphic to $c * (S^{n-1}, Y)$. This is done in the following two lemmas, which provide the basic procedure for the proof of Proposition (1).

LEMMA (4). *Let h be a homeomorphism mapping a pair P_1 to $c * (S^{n-1}, Y)$, and ϕ be a homeomorphism mapping $b * (a_1 \cup a_2) * (S^{n-2}, Z)$ to another pair P_2. Suppose that P_1 and P_2 intersect in the common sub-pair $\phi[a_1 * (S^{n-2}, Z)] \subset h^{-1}(S^{n-1}, Y)$. Then there exists a homeomorphism*

$$H : P_1 \cup P_2 \longrightarrow c * (S^{n-1}, Y).$$

If, further, for every point $z_i \in Z$, Y_i is defined as $h\phi(a_1\, z_i)$, H can be chosen so that for all z_i,

$$H^{-1}(cY_i) = h^{-1}(cY_i) \cup \phi[b * (a_1 \cup a_2) * z_i]$$

and $H^{-1}|c * (Y - \bigcup_i Y_i) = h^{-1}|c * (Y - \bigcup_i Y_i)$.

Proof. We shall define H and show that it has the required properties. Let P_3 be the pair $h^{-1}[c * h\phi(a_1 * (S^{n-2}, Z))]$. P_3 is a sub-pair of P_1, which meets the closure of $(P_1 - P_3)$ in $h^{-1}[c * h\phi(S^{n-2}, Z)]$, and $P_3 \cap P_2 = P_1 \cap P_2$. Define H to be equal to h when restricted to the closure of $(P_1 - P_3)$.

Now define $h_1 : P_3 \longrightarrow \tilde{c} * (a_1 * (S^{n-2}, Z))$ to be the composite of h followed by the conical extension of $[h\phi|a_1 * (S^{n-2}, Z)]^{-1}$. We have, given, a homeomorphism $\phi^{-1} : P_2 \longrightarrow b * (a_1 \cup a_2) * (S^{n-2}, Z)$ and, on $P_2 \cap P_3, h_1 = \phi^{-1}$. Thus h_1 and ϕ^{-1} together give a homeomorphism:

$$h_1 \cup \phi^{-1} : P_3 \cup P_2 \longrightarrow \tilde{c} * (a_1 * (S^{n-2}, Z)) \cup b\, G * (a_1 \cup a_2) * (S^{n-2}, Z).$$

If u is a point of $\tilde{c}\, a_1$ and v a point of $u\, a_1$, we can define a homeomorphism h_2,

$$h_2 : \tilde{c} * (a_1 * (S^{n-2}, Z)) \cup b * (a_1 \cup a_2) * (S^{n-2}, Z) \longrightarrow \tilde{c} * (a_1 * (S^{n-2}, Z))$$

by defining $h_2 | \tilde{c} * (S^{n-2}, Z)$ as the identity, $h_2 a_1 = u$, $h_2(b) = v$, $h_2(a_2) = a_1$ and extending linearly. Then $H | P_3 \cup P_2$ is defined to be $h\, h_1^{-1} h_2 (h_1 \cup \phi^{-1})$.

Now $h_1(h^{-1}[c * h\phi(S^{n-2}, Z)]) = \tilde{c} * (S^{n-2}, Z)$ and h_2 is the identity on $\tilde{c} * (S^{n-2}, Z)$. Thus on $P_3 \cap$ closure $(P_1 - P_3)$, $h_1^{-1} h_2 h_1$ is the identity, and H is a well defined homeomorphism on $P_1 \cup P_2$.

Let z_i be a point of Z; by definition $Y_i = h\phi(a_1 z_i)$. Then,

$$\begin{aligned} H^{-1}(cY_i) &= (h_1 \cup \phi^{-1})^{-1} h_2^{-1} h_1 h^{-1}(cY_i) \\ &= (h_1 \cup \phi^{-1})^{-1} h_2^{-1}(\tilde{c} a_1 z_i) \\ &= (h_1 \cup \phi^{-1})^{-1}(\tilde{c} a_1 z_i \cup b * (a_1 \cup a_2) * z_i) \\ &= h^{-1}[c * h\phi a_1 z_i] \cup \phi[b * (a_1 \cup a_2) * z_i] \\ &= h^{-1}(cY_i) \cup \phi[b * (a_1 \cup a_2) * z_i]. \end{aligned}$$

LEMMA (5). *Let h be a homeomorphism mapping a pair P_1 to $c * (S^{n-1}, Y)$, and ψ be a homeomorphism mapping $b * (B_1^{n-1} \cup B_2^{n-1}, g((a_1 \cup a_2) * Z))$ to another pair P_2, where $\partial B_1^{n-1} = \partial B_2^{n-1}$, $\dim Z \leqq n - 5$, and g is a proper embedding of $(a_1 \cup a_2) * Z$ in $B_1^{n-1} \cup B_2^{n-1}$. Suppose the intersection of P_1 and P_2 is the common subpair $\psi(B_1^{n-1}, g(a_1 Z)) \subset h^{-1}(S^{n-1}, Y)$. Then Theorem $(1^{(n-1)})$ implies there is a homeomorphism H,*

$$H : P_1 \cup P_2 \longrightarrow c * (S^{n-1}, Y),$$

such that, defining $Y_i = h\psi g(a_1 z_i)$ for each point $z_i \in Z$,

$$H^{-1}(cY_i) = h^{-1}(cY_i) \cup \psi[b * g((a_1 \cup a_2) * z_i)]$$

$$H^{-1} | c * (Y - \bigcup_i Y_i) = h^{-1} | c * (Y - \bigcup_i Y_i).$$

Proof. By Lemma (3) there is a homeomorphism θ, fixed on S,

$$\theta : (B_1 \cup B_2, g((a_1 \cup a_2) * Z)) \longrightarrow (a_1 \cup a_2) * (S, gZ)$$

where $S = \partial B_1 = \partial B_2$, such that $\theta g = \bar{g}$. This extends conically to

$$\bar{\theta} : b * (B_1^{n-1} \cup B_2^{n-1}, g((a_1 \cup a_2) * Z)) \longrightarrow b * (a_1 \cup a_2) * (S, gZ).$$

Then Lemma (4) gives at once the required result, using $\psi\bar{\theta}^{-1}$ as the homeomorphism ϕ of that lemma.

The following elementary result will also be required:

LEMMA (6). *Let B^n be an n-ball, containing another n-ball A, such that $\partial B^n \cap A$ is an $(n-1)$-ball F. Suppose that B_1^n, A_1 and F_1 are similarly described, and that h is a homeomorphism, $h : \partial B \longrightarrow \partial B_1$, such that $hF = F_1$, then can h be extended to a homeomorphism $\bar{h} : B \longrightarrow B_1$ such that $hA = A_1$.*

Proof. As mentioned in the proof of the corollary to Theorem (2), any homeomorphism of the boundary of a ball to the boundary of another extends to a homeomorphism between the two balls.

Define $\bar{h}|\partial B$ equal to h. Now $\partial A - \dot{F}$ is an $(n-1)$-ball, and we have defined already $\bar{h}|\partial(\partial A - \dot{F})$. Thus $\bar{h}$ extends over $\partial A - \dot{F}$, and we have $\bar{h} : \partial A \longrightarrow \partial A_1$. $\bar{h}$, being defined on ∂A, extends over A, mapping A to A_1. Similarly $\bar{h}$ is already defined on the boundary of the closure of $B - A$, and thus extends to the whole of B^n.

Combinatorial lemmas

Up to this point we have worked in the polyhedral category, but to proceed further we must consider the combinatorial category consisting of finite simplicial complexes and piecewise linear maps. It is to be assumed that all embeddings and homeomorphisms mentioned are in fact piecewise linear unless otherwise stated (occasionally simplicial maps will be required). The notation used for the join of two complexes, or for a pair of complexes (i.e. a complex and a subcomplex) will be the same as that used for polyhedra. If L is a subcomplex of K, $N(L, K)$ is the *closed simplicial neighbourhood* of L in K, being the subcomplex of K consisting of all simplexes that have a face in L, and of all faces of these simplexes. The subcomplex of $N(L, K)$ consisting of those simplexes not having a face in L is written $\partial N(L, K)$. If v is a vertex of K, we define, in the customary way, $\mathrm{star}(v, K) = N(v, K)$ and $\mathrm{link}(v, K) = \partial N(v, K)$. A Greek letter α, β, γ or δ before the symbol for a complex denotes some subdivision of that complex. K' denotes the first derived barycentric subdivision of K, and K'' the barycentric second derived. The polyhedron associated with K is written as $|K|$. Also, if a simplex σ is a face of another simplex τ, we denote this by $\sigma < \tau$.

LEMMA (7). *Let αK be a subdivision of the complex K, and let v be a vertex of αK. Then, if $v \in |\dot{\sigma}|$ for some simplex $\sigma \in K$, there is a homeomorphism h,*

$$h : \mathrm{star}(\sigma, K) \longrightarrow \mathrm{star}(v, \alpha K)$$

and for every simplex $\tau \in \mathrm{star}(\sigma, K)$, $h\tau = \mathrm{star}(v, \alpha\tau)$.

Proof. (i) Suppose first that v is also a vertex of K, i.e. that σ is 0-dimensional. Let $f : K \longrightarrow I$ be the unique simplicial map defined by:

$$fv = 0, fx = 1 \quad \text{for all other vertices } x \text{ in } K.$$

$|\mathrm{Star}(v, K)|$ can be parametrised by (l, t) where $l \in |\mathrm{link}(v, K)|$, $t \in [0, 1]$, so that if the point x has parameters (l, t), x lies on the line lv, and $fx = t$.

Let $f_\varepsilon : \mathrm{star}(v, K) \longrightarrow \mathrm{star}(v, K)$ be defined by $f_\varepsilon a = (a, \varepsilon)$, for all vertices a of $\mathrm{link}(v, K)$ $f_\varepsilon v = v$, the definition being completed by linearity. f_ε is a linear embedding, and $f_\varepsilon|\tau| \subset |\tau|$, for all $\tau \in \mathrm{star}(v, K)$. We choose ε sufficiently small, so that $|f_\varepsilon\, \mathrm{star}(v, K)| \subset |\overset{\circ}{\widehat{\mathrm{star}}}(v, \alpha K)|$. Let $\alpha_1 K$ be a subdivision of αK formed by making an elementary subdivision of each simplex of αK, in an order of decreasing dimension, choosing subdivision points on $|f_\varepsilon\, \mathrm{link}(v, K)|$ whenever possible. The image of f_ε is then a subcomplex of $\alpha_1 K$, namely star$(v, \alpha_1 K)$. But, as $\alpha_1 K$ is just a first derived of αK, there is a simplicial homeomorphism $g : \text{star}(v, \alpha_1 K) \longrightarrow v * (\text{link}(v, \alpha K))'$, such that if $\tau \in \text{star}(v, \alpha K)$, $g|\text{star}(v, \alpha_1 K) \cap \tau| = |\tau|$. Then $g f_\varepsilon$ gives the required homeomorphism h.

(ii) Suppose v is not a vertex of K. Let βK be the subdivision of K formed by an elementary subdivision at v. Let γK be a common subdivision of αK and βK. By (i), there is a homeomorphism $\phi : \text{star}(v, \beta K) \longrightarrow \text{star}(v, \gamma K)$ such that if $\tau \in \text{star}(v, \beta K)$, $\phi\tau = \text{star}(v, \gamma\tau)$, and there is a homeomorphism $\psi : \text{star}(v, \alpha K) \longrightarrow \text{star}(v, \gamma K)$ such that $\psi\tau = \text{star}(v, \gamma\tau)$ for all $\tau \in \text{star}(v, \alpha K)$. As $v \subset \dot{\sigma}$ for some $\sigma \in K$, let i be the identity map which maps star(σ, K) to star$(v, \beta K)$. Let h be $\psi^{-1}\phi i$. Then h is a homeomorphism, $h : \text{star}(\sigma\ K) \longrightarrow \text{star}(v, \alpha K)$ and if $\tau \in \text{star}(\sigma, K)$, $h\ \tau = \text{star}(v, \alpha\tau)$.

LEMMA (8). *Suppose Y^p is a combinatorial p-sphere containing a p-ball Z, and X is any complex. Let N be a subcomplex of a subdivision of XY, such that $N \cap Y$ is a p-ball, and for each simplex A^q of X, $N \cap A^q Y$ is a $(p + q + 1)$-ball. Then there is a homeomorphism $h : XY \longrightarrow XY$ such that*

$$hY = Y, \qquad hAY = AY \textit{ for all } A \in X, \qquad \textit{and}$$
$$h(XZ) = N.$$

Proof. Suppose X is of dimension x, and that X^r is the r-skeleton of X. We define, by induction on r, a homeomorphism $h_r : X^r Y \longrightarrow X^r Y$ with the properties: $h_r Y = Y$, $h_r A Y = A Y$ for all $A \in X^r$, and $h_r(AZ) = N \cap A Y$, for all $A \in X^r$.

First, define $h_{-1} : Y \longrightarrow Y$ to be any homeomorphism such that $h_{-1}Z = N \cap Y$. Then assume we have defined h_{r-1}, and that A is an r-simplex of X. $A Y$ is a $(p + r + 1)$-ball, with boundary $\partial(A Y) = \partial A * Y$. Now h_{r-1} is defined on $\partial(A Y)$, and $h_{r-1}^{-1}[N \cap (\partial A * Y)] = \partial A * Z$ which is a $(p + r)$-ball; $N \cap A Y$ is by definition a $(p + r + 1)$-ball. Thus $h_{r-1}|\partial A * Y$ is a homeomorphism of $\partial(A Y)$ to itself, sending $\partial A * Z$ to $N \cap \partial(A Y)$. By Lemma (6), $h_{r-1}|\partial A * Y$ can be extended to $h_r|A * Y$, such that $h_r|A Y : A Y \longrightarrow A Y$ and $h_r|A Y(AZ) = N \cap A Y$. This process, being repeated for all r-simplexes of X defines h_r with the required properties, and h_x is the required homeomorphism h.

Note. If L is a subcomplex of K, and (K, L) is equal to (XY, Y) in the notation of the above lemma, and if C is a collapsible subcomplex of L, $N(C'', K'')$ has the correct properties for N. Then the homeomorphism h, defined in the lemma has the additional property that $h(X * \partial Z) = \partial N(C'', K'')$.

The main proposition

We shall now state Proposition (1) and prove it with the aid of the lemmas developed in the section so far. Use will be made of the concept of simplicial collapsing as introduced by Whitehead [8]. We shall also need the notation of a conical subdivision of a cone as given in [10]:

If aK is the cone on a complex K, and $L \subset aK$, the subcone through L is the smallest polyhedron of $|aK|$ containing L of the form $|aJ|$ for $J \subset K$. A subdivision γaK of aK is called *conical* if the subcone through each simplex of γaK is a subcomplex of γaK. In particular, this subdivision induces a subdivision γK of K, and if $\tau \in \gamma K$, $\gamma a\tau$ is a subcomplex of γaK.

PROPOSITION (1). *Suppose $f: aK \longrightarrow B^n$ is a proper embedding of the cone on the complex K in the combinatorial ball B^n, with* $\dim(aK) \leqq n-3$. *There exists Y, a complex which subdivides B^n, and γaK a conical subdivision of aK, such that f maps γaK simplicially onto X, a subcomplex of Y. Let $fa = x$, then Theorem* $(1^{(n-1)})$ *implies that there is a homeomorphism*

$$h: [N(X'', Y''), X''] \longrightarrow [\mathrm{star}(x, Y''), \mathrm{star}(x, X'')]$$

such that for every $\tau \in \gamma K$, $hf(\gamma a\tau) = \mathrm{star}(x, f(\gamma a\tau)'')$.

Proof. Because f is piecewise linear, there are subdivisions of aK and B^n with respect to which f is simplicial. However, any subdivision of aK can be further subdivided to give a conical subdivision γaK, [10, Lemma (21)], and B^n can then be further subdivided so that Y is a subdivision of B^n and f maps γaK simplicially onto X, a subcomplex of Y. Thus X, Y and γaK always exist.

The cone $a(\gamma K)$ collapses simplicially to the vertex a, and the collapsing sequence can be chosen so that each elementary simplicial collapse has the effect of removing pairs of simplexes of the form $a\tau \cup \tau$, where $\tau \in \gamma K$, $a\tau_1$ being removed before $a\tau_2$, if $\dim \tau_1 > \dim \tau_2$. Because γ is a conical subdivision, if $\tau \in \gamma K$, $\gamma(a\tau)$ collapses simplicially to $\gamma(a\partial\tau)$, and in this process τ is removed in the first elementary collapse. Thus we can choose a collapsing sequence which collapses X to x, simplicially, which collapses away the whole of $fa\tau_1$ before any of $fa\tau_2$ if $\tau_i \in \gamma K$ and $\dim \tau_1 > \dim \tau_2$, and which only removes a simplex of $f\gamma K$ when it simultaneously removes another simplex having this first simplex as a face. Let this collapsing sequence be

$$X = X_k \searrow X_{k-1} \searrow \cdots \searrow X_0 = x.$$

We shall prove inductively the following statement, $\chi_{(i)}$:
'There is a homeomorphism $h_i: [N(X_i'', Y''), N(X_i'', X'')] \longrightarrow [\mathrm{star}(x, Y''), \mathrm{star}(x, X'')]$ such that $h_i(N(X_i'', X'') \cap f\gamma a\tau) = \mathrm{star}(x, f(\gamma a\tau)'')$ for all $\tau \in \gamma K$.'
$\chi(0)$ is trivially true, h_0 being the identity, and $\chi(k)$ is the statement of the proposition. Thus, inductively, assume that $\chi(i-1)$ is true.

Let $X_i = X_{i-1} + A + B$, A being an r-dimensional simplex, $A = cB$, and $X_{i-1} \cap A = c(\partial B)$; i.e. the elementary collapse $X_i \searrow X_{i-1}$ is performed by removing A and B. If $\hat{A}$ and $\hat{B}$ are the barycentres of A and B, then $[N(X_i'', Y''), N(X_i'', X'')] = [N(X_{i-1}'', Y''), N(X_{i-1}'', X'')] + [N(\hat{A}, Y''), N(\hat{A}, X'')] + [N(\hat{B}, Y''), N(\hat{B}, X'')]$. We must investigate the terms on the right hand side and determine how these three pairs glue together. We call these pairs P_1, P_2 and P_3 respectively.

(1) We first investigate $P_1 \cup P_2$:

$$P_2 = \hat{A} * [\mathrm{link}(\hat{A}, Y''), \mathrm{link}(\hat{A}, X'')].$$

Let $\mathrm{link}(A, Y) = S$, an $(n-r-1)$-combinatorial sphere since $\dot{A} \subset \dot{Y}$ from the selection of the collapsing sequence of X, and let $\mathrm{link}(A, X) = L$. Projecting the vertices of P_2, radially from $\hat{A}$ to $(\partial A)S$ gives a simplicial homeomorphism

$$H_1: P_2 \longrightarrow \hat{A} * [(\partial A'S')', (\partial A'L')'].$$

H_1 is the identity on $\hat{A}$, and any simplex of P_2 contained in $|A\sigma|$ for $\sigma \in S$, has its image under H_1 in $|A\sigma|$. H_1 also has the property that

$$H_1(P_1 \cap P_2) = N((c\partial B)'', (\partial A'S')'), N((c\partial B)'', (\partial A'L')').$$

Now, ∂A is an $(r-1)$-sphere, $N((c\partial B)'', \partial A'')$ is an $(r-1)$-ball Q, and for $\sigma' \in S$, $N((c\partial B)'', (\partial A'\sigma')')$ is an $(r+t)$-ball, because $c\partial B$ is collapsible. Thus, by Lemma (8), there is a homeomorphism

$$\theta : \partial AS \longrightarrow \partial AS$$

such that $\theta\partial A = \partial A$, $\theta Q = Q$, $\theta\partial A\sigma = \partial A\sigma$ for all $\sigma \in S$, (hence $\theta\partial AL = \partial AL$) and

$$\theta(Q * S) = N((c\partial B)'', (\partial A'S')').$$

There is a homeomorphism from $(v_1 \cup v_2) * S^{r-2}$ to ∂A, sending $v_1 S^{r-2}$ to Q, and this extends joinwise to a homeomorphism of $(v_1 \cup v_2) * S^{r-2} * S$ to ∂AS. Following this by θ, extending conewise and then following with H_1^{-1}, we obtain a homeomorphism

$$\phi_1 : u * (v_1 \cup v_2) * (S^{r-2}S, S^{r-2}L) \longrightarrow P_2$$

such that
$$\phi_1[v_1 * (S^{r-2}S, S^{r-2}L)] = P_1 \cap P_2$$

and for all $\sigma \in S$, $\phi_1[u * (v_1 \cup v_2) * S^{r-2}\sigma] = \text{star}(\hat{A}, Y'') \cap A\sigma$.

We are now in a position to apply Lemma (4). Applying this lemma to the homeomorphisms h_i and ϕ_1, we obtain the homeomorphism

$$\bar{h}_i : P_1 \cup P_2 \longrightarrow \text{star}(x, Y''), \text{star}(x, X'') = x * (\text{link}(x, Y''), \text{link}(x, X'')).$$

Now if $\tau \in \gamma K$,
$$\begin{aligned}\text{star}(\hat{A}, X'') \cap f(\gamma a\tau)'' &= \bigcup_{\sigma\in\Sigma} \text{star}(A, X'') \cap (A\sigma)'' \\ &= \bigcup_{\sigma\in\Sigma} \phi_1[u * (v_1 \cup v_2) * S^{r-2}\sigma]\end{aligned}$$
where $\Sigma = L \cap f(\gamma a\tau)$

and $\bigcup_{\sigma\in\Sigma} \phi_1(v_1 * S^{r-2}\sigma) \subset h_i^{-1}\,\text{link}(x, (fa\tau)'')$. Thus by Lemma (4), $\bar{h}_i$ can be chosen to have the additional property for all $\tau \in \gamma K$,

$$\bar{h}_i[(N(X_i, X'') \cup \text{star}(\hat{A}, X'')) \cap f(\gamma a\tau)''] = \text{star}(x, f(\gamma a\tau)'').$$

(2) We must now investigate $(P_1 \cup P_2) \cup P_3$. This must be a little more complicated than the preceding discussion, as it is here necessary to use Theorem $(1^{(n-1)})$.

$$P_3 = \hat{B} * [\text{link}(\hat{B}, Y''), \text{link}(\hat{B}, X'')]$$

If $T = \text{link}(B, Y)$ and $U = \text{link}(B, X)$, there is a simplicial homeomorphism

$$H_2 : P_3 \longrightarrow \hat{B} * ((\partial B' * T')', (\partial B' * U')')$$

where $H_2\hat{B} = \hat{B}$, and for any $\sigma \in T$, $|H_2^{-1}B\sigma| \subset |B\sigma|$. H_2 also has the property that

$$H_2((P_1 \cup P_2) \cap P_3) = N((\partial B' * c)', (\partial B' * T')'), N((\partial B' * c)', (\partial B' * U')').$$

From the choice of the collapsing sequence, there is some $\rho \in \gamma K$ such that $f(\dot{\widehat{\gamma a\rho}}) \supset \dot{A}$. Let $\Lambda = \text{link}(\rho, \gamma K)$. As f is a simplicial map from γaK to X, by Lemma (7) there is a homeomorphism

$$g : a\rho\Lambda \longrightarrow \text{star}(\hat{B}, X'')$$

such that $g(a\rho\tau) = \text{star}(\hat{B}, f(\gamma a\rho\tau)'')$ for all $\tau \in \Lambda$ including $\tau = \varnothing$. There are two possibilities to consider for B.

$$\text{either (i) } f(\widehat{\gamma a \rho}) \supset \mathring{B}$$

$$\text{or (ii) } f\,\dot{\rho} \supset \dot{B}$$

In case (ii), $\text{link}(\hat{B}, Y'')$ is an $(n-1)$-ball, g maps the cone $a\partial\rho\Lambda$ onto $\text{link}(\hat{B}, X'')$, and g gives a proper embedding of this cone in $\text{link}(\hat{B}, Y'')$. By Theorem $(1^{(n-1)})$ there is a homeomorphism ψ,

$$\psi : \text{link}(\hat{B}, Y''), \text{link}(\hat{B}, X'') \longrightarrow d * (\partial \text{ link}(\hat{B}, Y''), g(\partial\rho\Lambda))$$

which extends to a homeomorphism $\bar{\psi}$, by defining $\bar{\psi}\hat{B} = e$, and taking v as a point of e, d, where

$$\bar{\psi} : P_3 \longrightarrow v * (d \cup e) * (\partial \text{ link}(\hat{B}, Y''), g(\partial\rho\Lambda)).$$

Let $\partial P_3 = \bar{\psi}^{-1}[(d \cup e) * (\partial \text{ link}(\hat{B}, Y''), g(\partial\rho\Lambda))]$. Then $\bar{\psi}$ followed by a conical extension of $\bar{\psi}^{-1}|\bar{\psi}\partial P_3$ gives a homeomorphism

$$\xi : P_3 \longrightarrow v * \partial P_3.$$

By its construction ξ has the property that, if $\tau \in \Lambda$, $\xi g(a\rho\tau) = v * g(\partial a\rho)\tau$. In case (i), where P_3 is obviously just the cone on its boundary pair, we can define v as $\hat{B}$, and ξ as the identity. Then ξ has the same property as above, with respect to g, and we are now able to treat both cases simultaneously.

Let the boundary of P_3 be considered as the union of the two pairs (Q_1, R_1) and (Q_2, R_2), where $(Q_1, R_1) = P_3 \cap (P_1 \cup P_2)$, and (Q_2, R_2) is the closure of the compliment of this in ∂P_3. $N((\partial B' * c)', (\partial B' * T')')$ is an $(n-1)$-ball, being a regular neighbourhood of $\partial B' * c$ in the ball or sphere $(\partial B' * T')')$, and so Q_1, the image of this under H_2^{-1}, is also a ball. This implies that Q_2 is a ball. Suppose $\tau \in \Lambda$ or $\tau = \varnothing$, then

$$g((\partial a\rho)\tau) = (R_1 \cup R_2) \cap |f(a\rho\tau)|.$$

$$\begin{aligned} R_1 \cap f(\gamma a\rho\tau)'' &= H_2^{-1}[N((\partial B' * c)', (\partial B' * U')') \cap |f(a\rho\tau)|] \\ &= H_2^{-1}[N((\partial B' * c)', (\partial B' * W')')] \end{aligned}$$

where $W = \text{link}(B, f(\gamma a\rho\tau))$ which is a ball, and so $R_1 \cap f(\gamma a\rho\tau)''$ is a ball. Thus $R_1 \cap g((\partial a\rho)\tau)$ is a ball for all $\tau \in \Lambda$. Using Lemma (8) we can find a new homeomorphism

$$g' : ((\partial a\rho)\Lambda) \longrightarrow R_1 \cup R_2$$

such that $g'(a\partial\rho * \Lambda) = R_1$ and $g'(\partial a\rho)\tau = (R_1 \cup R_2) \cap f(a\rho\tau) = g(\partial a\rho)\tau$ for all $\tau \in \Lambda$ or $\tau = \varnothing$. Regarding $\partial(a\rho)$ as $(a_1 \cup a_2) * \partial\rho, g'$ gives a proper embedding of $(a_1 \cup a_2) * \partial\rho\Lambda$ in $Q_1 \cup Q_2$. Thus ξ^{-1} is a homeomorphism mapping $v * (Q_1 \cup Q_2, g'((a_1 \cup a_2) * \partial\rho\Lambda))$ to the pair P_3, and $(P_1 \cup P_2) \cap P_3 = \xi^{-1}(Q_1, g'(a_1 * \partial\rho\Lambda))$. Thus, applying Lemma (5) to the homeomorphisms ξ^{-1} and $\bar{h}_i$, we obtain the homeomorphism

$$h_{i+1} : P_1 \cup P_2 \cup P_3 \longrightarrow \text{star}(x, Y''), \text{star}(x, X'').$$

Further, for all $\tau \in \Lambda$, $\xi^{-1}(v * g'((a_1 \cup a_2) * \partial\rho\tau)) = \xi^{-1}(v * g(\partial a\rho)\tau) = g(a\rho\tau)$

$$= \text{star}(\hat{B}, f(\gamma a\rho\tau)'').$$

Thus, by Lemma (5), h_{i+1} can be chosen so that for $\tau \in \Lambda$, or $\tau = \varnothing$,

$$h_{i+1}[\mathrm{star}(\hat{B}, f(\gamma a \rho \tau)'') \cup \bar{h}_i^{-1} \mathrm{star}(x, f(\gamma a \rho \tau)'')]$$

$$= \mathrm{star}(x, f(\gamma a \rho \tau)''),$$

and if $\tau \in \gamma K$ and $\tau \notin \mathrm{star}(\rho, \gamma K)$,

$$h_{i+1} = \bar{h}_i \text{ on } \bar{h}_i^{-1} \mathrm{star}(x, f(\gamma a \tau)).$$

Hence h_{i+1} is the required homeomorphism fulfilling the conditions of the induction statement $\chi(i+1)$. Hence $\chi(i)$ implies $\chi(i+1)$, and the proposition is proven.

§4. SUNNY COLLAPSING AND PROOF OF THEOREMS

The Zeeman–Stallings technique of sunny collapsing is now required. This has been explained in [9]. Lemmas (7), (8) and (9) of that paper are needed here, but in a sharpened form. The improved versions are given below at Lemmas (10), (9) and Proposition (2); proofs (modelled on those of [9]) of these lemmas are given, particular attention being paid to any points where a proof differs from being a paraphrase of the analogous lemma in [9].

Let I^p be the p-cube, then $I^p = I^{p-1} \times I$, and we can regard I^{p-1} as horizontal and I as vertical. If X is a polyhedron in I^p, let X^* be the set of points of X that lie in the same vertical line as some other point of X. Then $\bar{X}^*$, the closure of X^*, is a subpolyhedron of X.

LEMMA (9). *Let X_0 be a complex linearly embedded in I^n, with* $\dim X_0 \leqq n-3$ *and* $\dim (X_0 \cap \partial I^n) \leqq n-4$. *Then (I^n, X_0) is homeomorphic to (I^n, X) where X is isomorphic to a subdivision of X_0, such that*

(i) *X does not meet the top or bottom of the cube;*

(ii) *X meets any vertical line finitely.*

Furthermore, if αX is a subdivision of X such that the vertical projection $\pi : \alpha X \longrightarrow I^{n-1}$ is simplicial for some subdivision of I^{n-1}, and if L is the subcomplex of αX triangulating $\bar{X}^*$ then, defining A_τ to be the simplex of X such that $\dot{\tau} \subset \dot{A}_\tau$ for every $\tau \in L$,

(iii) If $\tau_1, \tau_2 \in L$ and $\pi\tau_1 = \pi\tau_2$, then $\dim \tau_i \leqq \dim A_{\tau_i} - 2$

(iv) If $\tau_1, \tau_2 \in L$, $\pi\tau_1 = \pi\tau_2$, $\sigma < \tau_1$ and $\dim \sigma > \dim A_\sigma - 2$, then $\sigma < \tau_2$.

Proof. Let I_1^{n-1} be a vertical $(n-1)$-dimensional face of I^n. First choose a homeomorphism of I^n to itself sending $X_0 \cap \partial I^n$ into $\dot{I}_1^{n-1}$. Triangulate I^n so that the image of X_0 is a subcomplex X, choosing X to be isomorphic to some subdivision of X_0. Now shift all the vertices of X by arbitrarily small moves into general position in the following way: Shift the vertices of $X \cap \dot{I}_1^{n-1}$ so that each vertex remains in $\dot{I}_1^{n-1}$, and no vertex lies in the vertical t-plane through any other t vertices, for $t \leqq n-2$; similarly shift the vertices in $\dot{I}^n$ so that the same condition holds for $t \leqq n-1$. If the moves are sufficiently small, they determine a homeomorphism of I^n to itself sending X to this new position, which gives the required pair (I^n, X).

It is clear that conditions (i) and (ii) are satisfied. If A^a and B^b are simplexes of X with no face in common, then the vertical $(a+1)$-plane containing A meets the plane of B in a plane of dimension $(a+b+1-n) \leqq b-2$, because $a \leqq n-3$. If however A^a and B^b intersect in a common face C^c, then the vertical $(a+1)$-plane through A meets the plane of B in a plane of dimension max $[c, (a+b+1-n)]$ which always contains C, but which intersects $\dot{B}$ in dimension $(a+b+1-n) \leqq b-2$, or not at all.

If αX is a subdivision of X as described in the statement of the lemma, suppose $\pi\tau_1 = \pi\tau_2$ for $\tau_i \in L$, then from the above discussion, $\dot{\tau}_i \subset \dot{B}_i$ for $B_i \in X$, and dim $B_i \geqq$ dim $\tau_i + 2$. If further $\sigma < \tau_1$, $\dot{\sigma} \subset A$ for $A \in X$, and dim $\sigma + 2 >$ dim A, then the vertical planes through each B_i contain σ and so both these planes meet $\dot{A}$ in dimension greater than dim $A - 2$. Thus A is a face of each B_i, and as $\pi\sigma < \pi\tau_2$, and $\tau_2 \subset B_2$, σ is a face of τ_2.

DEFINITIONS. *If X is a polyhedron in I^p, a point of I^p lies in* the shadow *of X if it is vertically below some point of X. Suppose $X \searrow Y$ is an elementary (polyhedral) collapse from X to Y* [10], *this collapse is called* sunny *if no point of $X - Y$ lies in the shadow of X. A sequence of elementary sunny collapses is called a* sunny collapse, *and if there is a sunny collapse from X to a point, X is said to be* sunny collapsible.

Let (X, Y) be a pair of polyhedra triangulated by the pair of complexes (K, L). The *local codimension of Y in X* is the maximum number r such that every t-simplex of L faces some $(t+r)$-simplex of K. The local codimension is independent of the particular triangulation (K, L).

PROPOSITION (2). *If $g : aY \longrightarrow B^n$ is a proper embedding of the cone on the polyhedron Y in B^n, with* dim $aY = m \leqq n-3$, *then $(B^n, g(aY))$ is homeomorphic to (I^n, X) where X satisfies the properties of Lemma* (9), *and X is sunny collapsible.*

Proof. First, by Lemma (9), choose a homeomorphism of $(B^n, g(aY))$ to (I^n, X) where X is a complex satisfying the properties of that lemma.

We shall construct inductively a decreasing sequence of subpolyhedra:

$$|X| = X_0 \supset X_1 \supset \dots \supset X_m = \text{a point},$$

and for each $i \leqq m$ a homeomorphism

$$f_i : X_i \longrightarrow vK^{m-i-1}$$

where K^{m-i-1} is some $(m-i-1)$-dimensional complex, such that;

(1) $f_i^{-1}v$ overshadows no point of X_i.

(2) If P and Q are arcs in X_i^* and P overshadows Q, and f_iP lies in a generator of vK^{m-i-1}, then so does f_iQ.

(3) The subcone of vK^{m-i-1} through $f_i\bar{X}_i^*$ is locally of codimension greater than or equal to one in vK^{m-i-1}.

(4) There is a sunny collapse $X_{i-1} \searrow X_i$.

The induction starts with $|X| = X_0$. Condition (4) is vacuous, and it is necessary to define the homeomorphism f_0 with properties (1), (2), and (3). Let K^{m-1} triangulate Y.

Using the notation of Lemma (9), the complex X is isomorphic to some subdivision of vK^{m-1}, thus let $f: X \longrightarrow vK^{m-1}$ be this homeomorphism. Thus in particular, f maps simplexes of L(N.B. L is the subcomplex of X triangulating $\bar{X}^*$) linearly into simplexes of vK^{m-1}. Now move the vertices of fL into general position in vK^{m-1} such that any vertex in $\widehat{vA}$ remains in $\widehat{vA}$ for every $A \in K^{m-1}$; and so that, in the new position, if $\tau \in L$ and $f\tau$ does not meet generators of vK^{m-1} finitely, then τ has some face σ, such that $f\sigma \subset vA$, and $\dim \sigma = \dim vA$. If the moves are chosen sufficiently small, there is a new homeomorphism $f_0 : \alpha X \longrightarrow vK^{m-1}$, such $f_0 L$ agrees with the new general position of fL, (f_0 embeds each simplex of L linearly in some simplex of vK^{m-1}).

If $\tau \in L$, Lemma (9), (iii), implies that τ is the face of some simplex $\rho \in \alpha X$, where $\dim \tau \leqq \dim \rho - 2$, so the cone on $f_0 L$ is locally of codimension $\geqq 1$ in vK^{m-1}. Condition (1) is satisfied as $f_0^{-1}v$ is a vertex of X, so it remains to consider (2). If $\tau_1 \in L$, and $f_0\tau_1$ does not meet generators of the cone vK^{m-1} finitely, τ_1 has a face σ such that $f\sigma \subset vA$ and $\dim \sigma = \dim vA$. By Lemma (9), (iv), if $\pi\tau_1 = \pi\tau_2$, τ_2 has σ as a face, so $f_0\tau_1$ and $f_0\tau_2$ meet in $f_0\sigma$, and so $f_0\tau_2$ meets generators of the cone non-finitely. There is a simplicial map $s : f_0\tau_1 \longrightarrow f_0\tau_2$, fixed on $f_0\sigma$, so that if $x \in |f_0\tau_1|$, $\pi f_0^{-1}x = \pi f_0^{-1}sx$. This extends to a *linear* map of the cone through $f_0\tau_1$ to the cone through $f_0\tau_2$, that is fixed on the subcone through $f_0\sigma$. Thus if A and B are two points in $f_0\tau_1$ such that the line AB passes through v, then the line $(sA)(sB)$ also passes through v. Thus if P and Q are polyhedral paths in L, such that $\pi P = \pi Q$, and if $f_0 P$ lies in a generator of vK, Then $f_0 Q$ also lies in a generator (for $f_0 P$ can be split up into a union of intervals as AB above) and so (2) is satisfied.

The induction finishes with X_m, a single point, so we shall have a sunny collapse

$$|X| \searrow X_1 \searrow X_2 \searrow \dots \searrow X_m$$

which will prove the proposition. The proof of the induction step is as follows:

Suppose we are given $f_{i-1} : X_{i-1} \longrightarrow vK^{m-1}$ satisfying the four conditions. We have to construct f_i, X_i, K^{m-i-1} and prove that the four conditions still hold.

Let (C, L) triangulate $(vK^{m-i}), f_{i-1}\bar{X}^*_{i-1})$. C can be chosen to be a conical subdivision of vK^{m-i}, and so, in particular C contains a subdivision γK^{m-i} of K^{m-i}. We now define K^{m-i-1} by

$$K^{m-i-1} = (m-i-1) - \text{skeleton of } \gamma K^{m-i}.$$

Let C_0 be the subcomplex of C triangulating the subcone vK^{m-i-1}. Let $e_0 : C_0 \longrightarrow C$ be the inclusion map. We shall construct a slightly different embedding $e : C_0 \longrightarrow C$. Having chosen e, there is uniquely defined a polyhedron X_i together with a homeomorphism f_i, such that the following diagram is commutative:

$$\begin{tikzcd} X_i \arrow[r, "\subset"] \arrow[d, "f_i"] & X_{i-1} \arrow[d, "f_{i-1}"] \\ C_0 \arrow[r, "e"] & C \end{tikzcd}$$

Now condition (3) implies that $L \subset C_0$, so $\dim L \leqq m - i$. If π is the vertical projection $\pi : \bar{X}^*_{i-1} \longrightarrow I^{n-1}$, let βL be a subdivision of L such that

$$\pi f_{i-1}^{-1} : \beta L \longrightarrow I^{n-1}$$

is simplicial for some subdivision of I^{n-1}. Let $A_1, A_2, \ldots, A_r$ be an ordering of the $(m-i)$-simplexes of βL, such that all points of X that overshadow $f_{i-1}^{-1}\dot{A}_k$ are contained in $\bigcup_{j<k} f_{i-1}^{-1}\dot{A}_j$. Similarly let $B_1, B_2, \ldots, B_s$ be an ordering of the $(m-i-1)$-simplexes of βL which cut generators of C finitely, such that all points of X that overshadow $f_{i-1}^{-1}\dot{B}_k$ are contained in $\bigcup_{j<k} f_{i-1}^{-1}\dot{B}_j$. This last ordering is possible as, by condition (3), if $\sigma, \tau \in \beta L$, and $f_{i-1}^{-1}\sigma$ overshadows $f_{i-1}^{-1}\tau$, then if σ cuts generators of C non-finitely, so does τ.

The next step is to construct little $(m-i-1)$-dimensional blisters Y_j about each A_j, and Z_j about each B_j in the cone C. These blisters are the necessary device for making the required sunny collapse. For each particular j, let $\hat{A}_j$ be the barycentre of A_j. Since the subcone through A_j is locally of codimension $\geqq 1$, there is some simplex $D^{m-i} \in \gamma K^{m-i}$ such that A_j is contained in the subcone vE where $E < D$. Let a_j be a point in $v\dot{D}$ near $\hat{A}_j$, and define the blister

$$Y_j = a_j A_j.$$

We choose the points a_j sufficiently near to the barycentres so that no two blisters meet more than is necessary, (i.e. $Y_j \cap Y_k = \partial A_j \cap \partial A_k$). The *bottom* of the blister A_j, and the *top* is $a_j \partial A_j$. For all $|A_k| \subset |vE|$, the blisters are chosen so that $a_k \in |v\dot{D}|$.

As C_0 meets each blister in its bottom we can define an embedding $e_1 : C_0 \longrightarrow C$ by defining e_1 as the map that, when restricted to the intersection with the blisters, sends the bottom of each blister to the top mapping $\hat{A}_j$ to a_j, and as the identity otherwise.

We now define the blisters Z_j about B_j. As in the above discussion, there is some $E^{m-i-1} \in \gamma K^{m-i}$, such that $B_j \subset vE$, and $E < D^{m-i} \in \gamma K^{m-i}$. If some A_i are contained in vE, D is chosen to be the simplex such that $|vD| \supset |Y_i|$. Let $\hat{B}_j$ be the barycentre of B_j. There are two cases to consider depending on whether or not B_j lies in E: If $|B_j| \subset |E|$, let b_j be a point in $\dot{D}$ near $\hat{B}_j$, and c_j a point in the generator of the cone $e_1 C_0$ (note: *not* of $e_0 C_0$) through $\hat{B}_j$, near to $\hat{B}_j$. Otherwise, if $|B_j| \not\subset |E|$, let b_j be a point in $v\dot{D}$ near $\hat{B}_j$, and c_j be a *pair* of points on the generator of the cone $e_1 C_0$ through $\hat{B}_j$, near $\hat{B}_j$, and on either side of $\hat{B}_j$. (The generators of the cone $e_1 C_0$ are used because then some points c_j will lie on the tops of some of the blisters Y_i already defined.) In either case define the blister

$$Z_j = c_j b_j B_j.$$

By choosing all the points sufficiently near the barycentres, $Z_j \cap Z_k = \partial B_j \cap \partial B_k$, the *bottom* of Z_j is $c_j B_j$ and the *top* is $c_j b_j \partial B_j$. If b_j is always chosen outside the blisters Y_i, $e_1 C_0$ meets the blisters Z_j in the bottoms only. Now define the required embedding $e : C_0 \longrightarrow C$ by modifying e_1 (as e_0 was previously modified to give e_1), so that e differs from e_1 only insomuch as it maps $e_1^{-1} c_j B_j$ to $c_j b_j \partial B_j$. Having defined e, we have completed the definition of X_i and of $f_i : X_i \longrightarrow vK^{m-i-1}$.

It remains to verify the four conditions. Condition (1) holds because $f_i^{-1} v = f_{i-1}^{-1} v$.

Condition (2) holds because $f_i X_i^* = f_{i-1} X_i^* \subset f_{i-1} X_{i-1}^*$ for which the condition holds by induction. If σ is a simplex of $\beta L \cap f_{i-1}\overline{X}_i^*$, the subcone $v\sigma$ is of dimension $t \leqq m-i-1$. As C is conical, there is some simplex D of K^{m-i-1}, such that $|v\sigma| \subset |vD|$. Either dim $D \geqq t$, or else, by the induction condition (3), D faces some simplex in γK^{m-i} of dimension t, and as $t \leqq m-i-1$, this simplex is in K^{m-i-1}. Therefore conditions (3) is satisfied.

Lastly, it must be shown that X_{i-1} sunny collapses to X_i. The proof of this is very similar to that in [9]. There is a collapse

$$C \searrow eC_0 \cup Y \cup Z$$

where $Y = \bigcup_i Y_i$, and $Z = \bigcup_i Z_i$. We then collapse away Y and Z by collapsing the blisters Y_i and Z_i from the bottom to the top in the order $Y_1, Y_2, \ldots, Y_{r-1}, Y_r, Z_1, Z_2, \ldots, Z_s$. The inverse image of this collapsing sequence under f_{i-1} gives a sunny collapse

$$X_{i-1} \searrow X_i.$$

This finishes the proof of the proposition, and we can at once deduce the following lemma:

LEMMA (10). *If $g : aY \longrightarrow B^n$ is a proper embedding, with* dim $aY \leqq n-3$, *then B^n collapses onto $g(aY)$.*

Proof. By Proposition (2), $(B^n, g(aY))$ is homeomorphic to (I^n, X) where X satisfies (i) and (ii) of Lemma (9), and X is sunny collapsible. Then by [9], there is a collapse $I^n \searrow X$. Hence also $B^n \searrow g(aY)$.

In the proof of the main theorems we shall also need the following simple lemma:

LEMMA (11). *Let L be a subcomplex of K, and f be a homeomophism of L to itself such that for all $\tau \in L$, $|f\tau| = |\tau|$. Then f can be extended to a homeomorphism $F : K \longrightarrow K$, such that for all $\sigma \in K$, $|F\sigma| = |\sigma|$, and if, for some $\sigma \in K, f|\sigma \cap L = 1$, then $F|\sigma = 1$.*

Proof. Let $\{\sigma_i\} i = 1, 2, \ldots, n$, be an enumeration of the simplexes of K in an order of increasing dimension, and let $K_i = \bigcup_{j=1}^{i} \sigma_j$. Suppose, inductively, that there is a homeomorphism $F_i : K_i \longrightarrow K_i$ such that $|F_i\sigma| = |\sigma|$ for all $\sigma \in K_i$, $F_i|K_i \cap L = f|K_i \cap L$, and if $f|\sigma \cap L = 1$ then $F_i|\sigma = 1$. Then extend F_i to F_{i+1} as follows:

$$F_{i+1}|\sigma_{i+1} = \begin{cases} f|\sigma_{i+1} & \text{if} \quad \sigma_{i+1} \in L, \\ 1 \text{ if } F_i|\partial\sigma_{i+1} = 1 \text{ and } \sigma_{i+1} \notin L, \\ \text{any extension of } F_i|\partial\sigma_{i+1} \text{ otherwise.} \end{cases}$$

We define $F_1 = 1$, and F_n is the required homeomorphism F.

Proof of Theorem (1)

Assuming Theorem $(1^{(n-1)})$ we have to prove Theorem $(1^{(n)})$. Thus let $f : aX \longrightarrow B^n$ be a proper embedding where dim $aX \leqq n-3$. Let K triangulate X; let γvK be a conical subdivision of vK, so that γvK triangulates aX, and so that for some triangulation M of $B^n, f : \gamma vK \longrightarrow M$ is simplicial. Let $f(\gamma vK) = L \subset M$.

Lemma (10) implies that M is a regular neighbourhood of L, so by [6], there is a homeomorphism of (M'', L'') to $(N(L'', M''), L'')$ that is the identity on L''. Thus, applying Proposition (1) (which assumes Theorem $(1^{(n-1)})$, there is a homeomorphism

$$h : (M'', L'') \longrightarrow [\text{star}(x, M''), \text{star}(x, L'')]$$

where $x = fv$, and such that for every $\tau \in \gamma K$

$$hf(\gamma v\tau) = \text{star}(x, f(\gamma v\tau)'').$$

The inverse homeomorphism of $h|\partial M''$ gives a homeomorphism

$$g : [\text{link}(x, M''), \text{link}(x, L'')] \longrightarrow (\partial M'', \partial M'' \cap L''),$$

and g extends conically to

$$\bar{g} : [\text{star}(x, M''), \text{star}(x, L'')] \longrightarrow w * (\partial M'', \partial M'' \cap L'').$$

Combining h and $\bar{g}$ we have

$$\bar{g}h : (M'', L'') \longrightarrow w * (\partial M'', \partial M'' \cap L)$$

with the properties that $\bar{g}h|(\partial M'', \partial M'' \cap L)$ is the identity, $\bar{g}hfv = w$, and if $\tau \in \gamma K$, $\bar{g}hf(\gamma v\tau) = w * f\tau$.

Now $f|X$ extends conically to $\bar{f} : aX \longrightarrow b * \partial B^n$. Then $v(\gamma K)$ triangulates aX, $w * \partial M$ triangulates $b * \partial B^n$, and the embedding

$$\bar{f} : v(\gamma K) \longrightarrow w * \partial M$$

is simplicial. Let C be the subcomplex $\bar{f}(v(\gamma K))$ of $w * \partial M$. Then $\bar{g}hf\bar{f}^{-1}$ is a homeomorphism of C to itself, which restricts to the identity on $C \cap \partial M$ and such that

$$|\bar{g}hf\bar{f}^{-1}\tau| = |\tau| \quad \text{for all} \quad \tau \in C.$$

By Lemma (11), $\bar{g}hf\bar{f}^{-1}$ extends to a homeomorphism

$$h_* : w * \partial M \longrightarrow w * \partial M, \quad \text{with } h_*|\partial M = 1.$$

Let $H = h_*^{-1}\bar{g}h$, then

$$H : (M, L) \longrightarrow (w * \partial M, w * (\partial M \cap L)),$$

$H|\partial M = 1$ and $Hf = h_*^{-1}\bar{g}hf = (\bar{g}hf\bar{f}^{-1})^{-1}\bar{g}hf = \bar{f}$. Hence reverting to polyhedra, we have found H such that

$$H : B^n \longrightarrow b * \partial B^n, \qquad H|\partial B^n = 1, \quad \text{and} \quad Hf = \bar{f}.$$

Thus Theorem $(1^{(n)})$ is proved, and this completes the inductive proof of Theorem (1).

Proof of Theorem (6)

The proof of Theorem (6) is closely related to that of Theorem (1), and now that Theorem (1) has been proved, Proposition (1) is known to be true for all n.

Let f and g be embeddings of the polyhedron X^p in S^n, where $n - p \geqq 3$. Let H be an embedding $H : X \times I \longrightarrow S^n \times I$ such that $H^{-1}(S^n \times 0) = X \times 0$, $H^{-1}(S^n \times 1) = X \times 1$, and $H(x, 0) = (fx, 0)$ and $H(x, 1) = (gx, 1)$. Let L triangulate $X \times I$ and K triangulate

$S^n \times I$ such that $H : L \longrightarrow K$ is simplicial. Further, choose L to be a *cylindrical* triangulation, i.e. If L_0 and L_1 are the subcomplexes of L triangulating $X \times 0$ and $X \times 1$, then the natural projections of L to L_0 and L_1 are simplicial (it is always possible to choose such an L, see [10]). Let K_0 and K_1 be the subcomplexes of K triangulating $S^n \times 0$ and $S^n \times 1$. We now add to L and K, cones on L_1 and K_1, and extend H conically to the embedding

$$\bar{H} : L \cup aL_1 \longrightarrow K \cup bK_1.$$

Then $\bar{H}$ is a simplicial embedding of $L \cup aL_1$, which is a conical subdivision of the cone vL_0 on L_0, to $K \cup bK_1$ which is a triangulation of the $(n+1)$-ball. Letting $C = \bar{H}(L \cup aL_1)$ and $M = K \cup bK_1$, Proposition (1) implies there is a homeomorphism

$$h_1 : [N(C'', M''), C''] \longrightarrow [\operatorname{star}(b, M''), \operatorname{star}(b, C'')]$$

Using Lemma (10) and the uniqueness of regular neighbourhood theory [6] as before, there is a homeomorphism

$$h_2 : [M'', C''] \longrightarrow [N(C'', M''), C'']$$

such that if $\tau \in L_0$,

$$h_1 h_2 \bar{H}(v\tau) = \operatorname{star}(b, \bar{H}(v\tau) \cap M'').$$

Now Lemma (7) gives a homeomorphism

$$h_3 : [\operatorname{star}(b, M''), \operatorname{star}(b, C'')] \longrightarrow [bK_1, b(\bar{H}L_1)]$$

such that for all $\sigma \in L_1$,

$$h_3 \operatorname{star}(b, \bar{H}(a\sigma) \cap M'') = b\sigma.$$

Combining all these homeomorphisms together, we obtain

$$\theta = h_3 h_1 h_2 : [M, C] \longrightarrow [bK_1, b(HL_1)].$$

θ has the property that if $\tau \in L_0$, $\theta\bar{H}(v\tau) = b\sigma$, where σ is the unique simplex of L_1, which projects onto τ in the natural simplicial projection of L_1 to L_0. Let $i : L_1 \longrightarrow L_0$ be this isomorphism. Restricting θ to ∂M, gives a homeomorphism $\partial\theta$,

$$\partial\theta : (K_0, L_0) \longrightarrow (K_1, L_1),$$

$(\partial\theta)i$ maps L_1 to itself, and $|(\partial\theta)i\sigma| = |\sigma|$ for all $\sigma \in L_1$. Using Lemma (11), there is a homeomorphism

$$F : (K_1, L_1) \longrightarrow (K_1, L_1)$$

such that $F(\partial\theta)i$ is the identity on L_1. Thus $F(\partial\theta)$ maps (K_0, L_0) to (K_1, L_1), and $F(\partial\theta)|L_0 = i^{-1}$.

But, L_0 triangulates X, K_0 triangulates S^n and $L_0 \subset K_0$ is the inclusion given by the embedding f; similarly, $L_1 \subset K_1$ represents g and the following diagram computes:

$$\begin{tikzcd}
L_0 \arrow[r, "\subset", "f"'] \arrow[d, "i^{-1}"] & K_0 \arrow[d, "F(\partial\theta)"] \\
L_1 \arrow[r, "\subset", "g"'] & K_1
\end{tikzcd}$$

Hence, reverting to polyhedra, $F(\partial\theta)$ maps S^n to S^n, and $F(\partial\theta)f = g$, and so f and g are ambient isotopic and the theorem is proved.

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